5.1 Area and Distance Problems

Learning Objectives: After completing this section, we should be able to

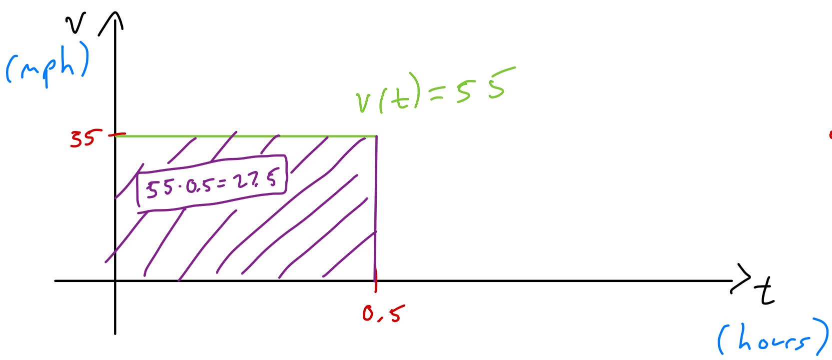
- find the area under a curve by estimating the sum of the areas of rectangular strips.
- apply finding the area under a curve in distance problems.

Example. Suppose I drove my car at a constant velocity of 55 MPH for 0.5 hours.

How far did I travel?

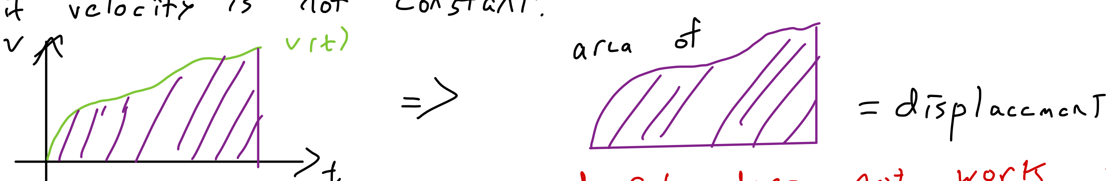
$$\text{distance} = (\text{rate})(\text{time})$$
$$= \left(55\ \frac{\text{miles}}{\text{hour}}\right)(0.5\ \text{hours})$$
$$= 27.5\ \text{miles}$$

How else can this be represented?



• area under the curve $v(t)=55$ gave the distance travelled

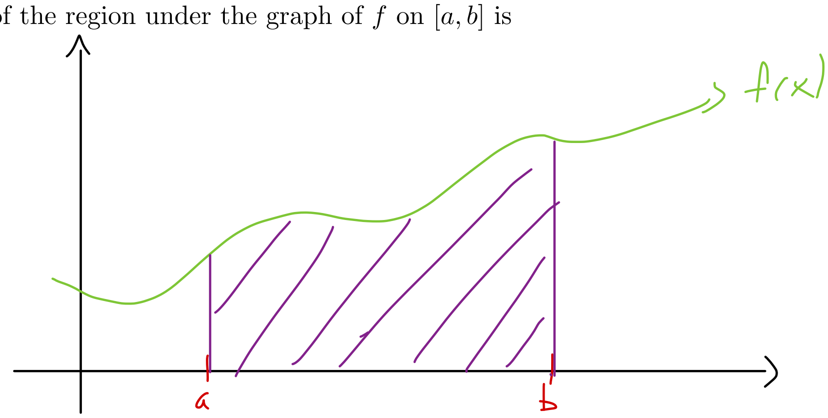
what if velocity is not constant?



Note, $d = r \cdot t$ does not work for non-constant rates

5.1.1 Area Under a Curve

The area of the region under the graph of f on $[a, b]$ is

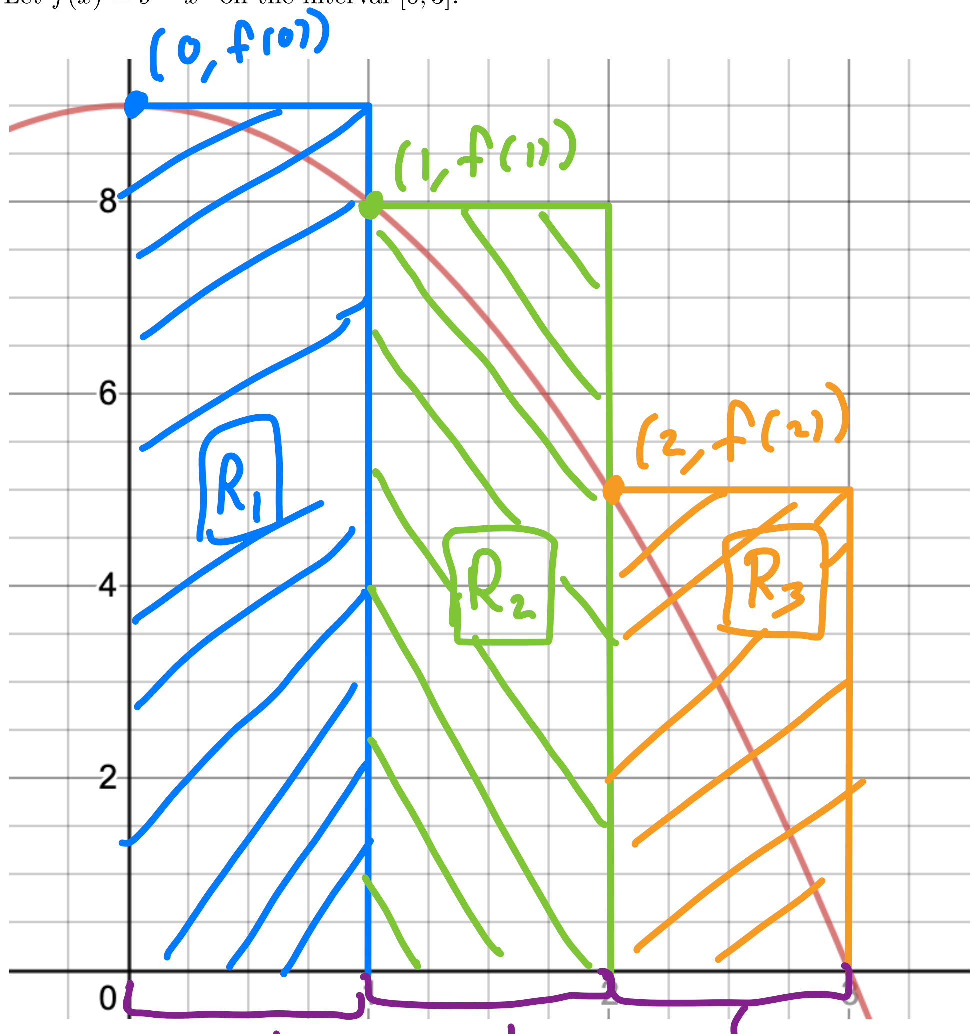


area of purple region is "area under the curve/graph of f on $[a, b]$".

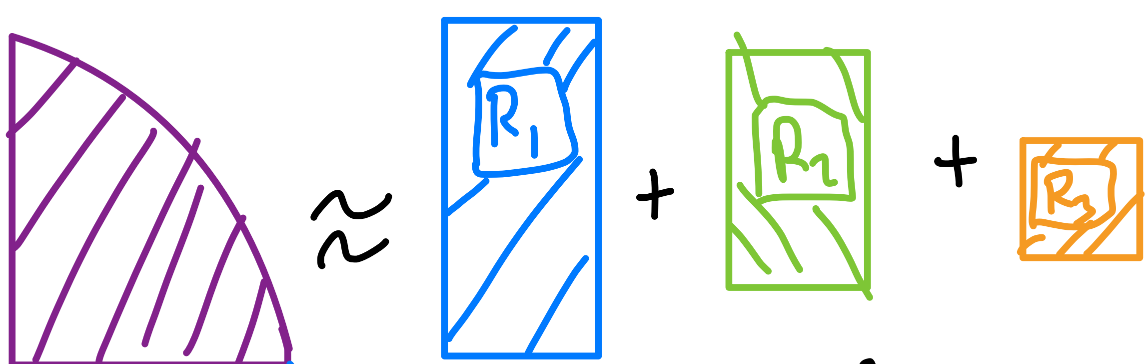
How can we approximate area under the curve?

Example. Let $f(x) = 9 - x^2$ on the interval $[0, 3]$.

(Left)



We have an 'area' formula for rectangles, so we use them to approximate area under a curve.



$R_1 = (\text{width})(\text{height}) = (1)(f(0)) = (1)(9 - 0^2) = 9$

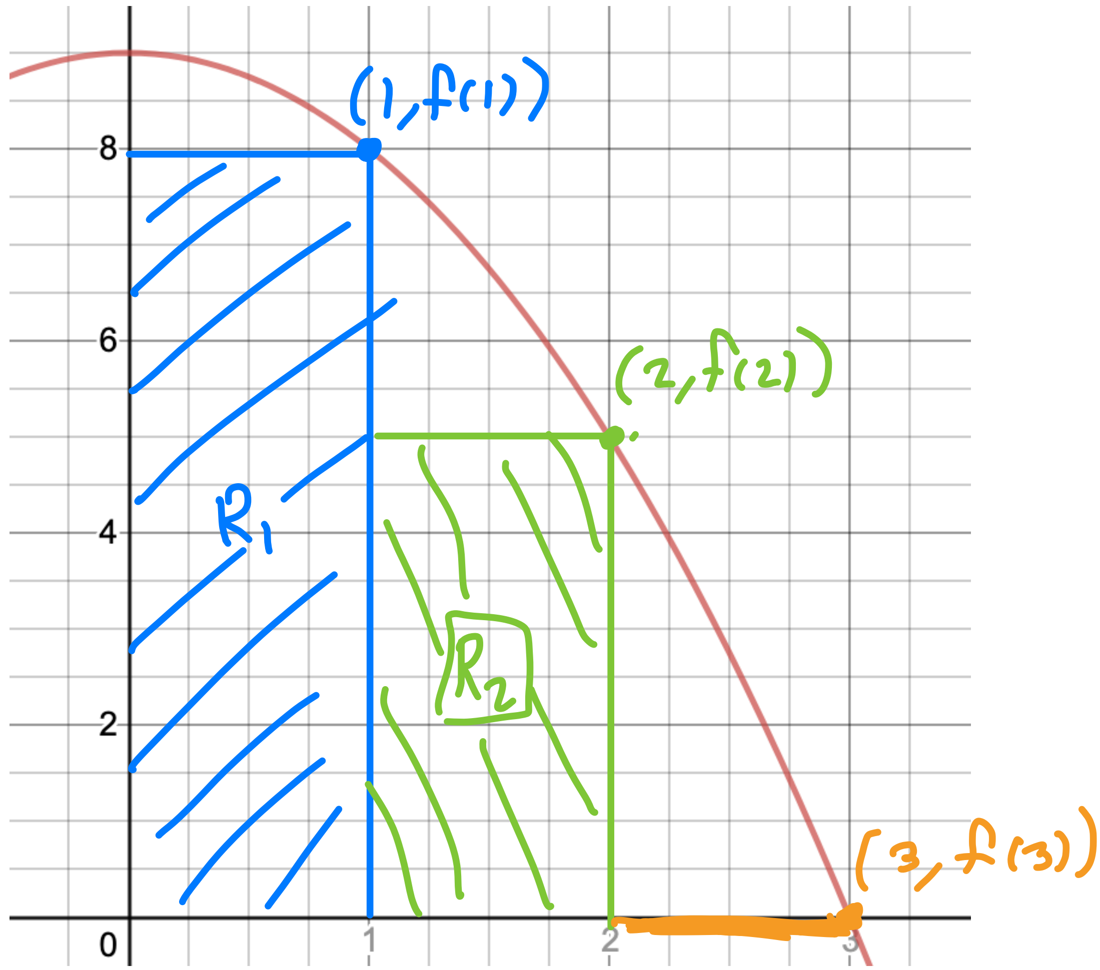
$R_2 = (\text{width})(\text{height}) = (1)(f(1)) = (1)(9 - 1^2) = 8$

$R_3 = (\text{width})(\text{height}) = (1)(f(2)) = (1)(9 - 2^2) = 5$

Estimated area = 9 + 8 + 5 = 22 (overestimate)

Let $f(x) = 9 - x^2$ on the interval $[0, 3]$. $f(x) = 9 - x^2$

(Right)



$R_1 = (\text{width})(\text{height}) = (1) \cdot f(1) = 1 \cdot (9 - 1^2) = 8$

$R_2 = (\text{width})(\text{height}) = (1) \cdot f(2) = 1 \cdot (9 - 2^2) = 5$

$R_3 = (\text{width})(\text{height}) = (1) \cdot f(3) = 1 \cdot (9 - 3^2) = 0$

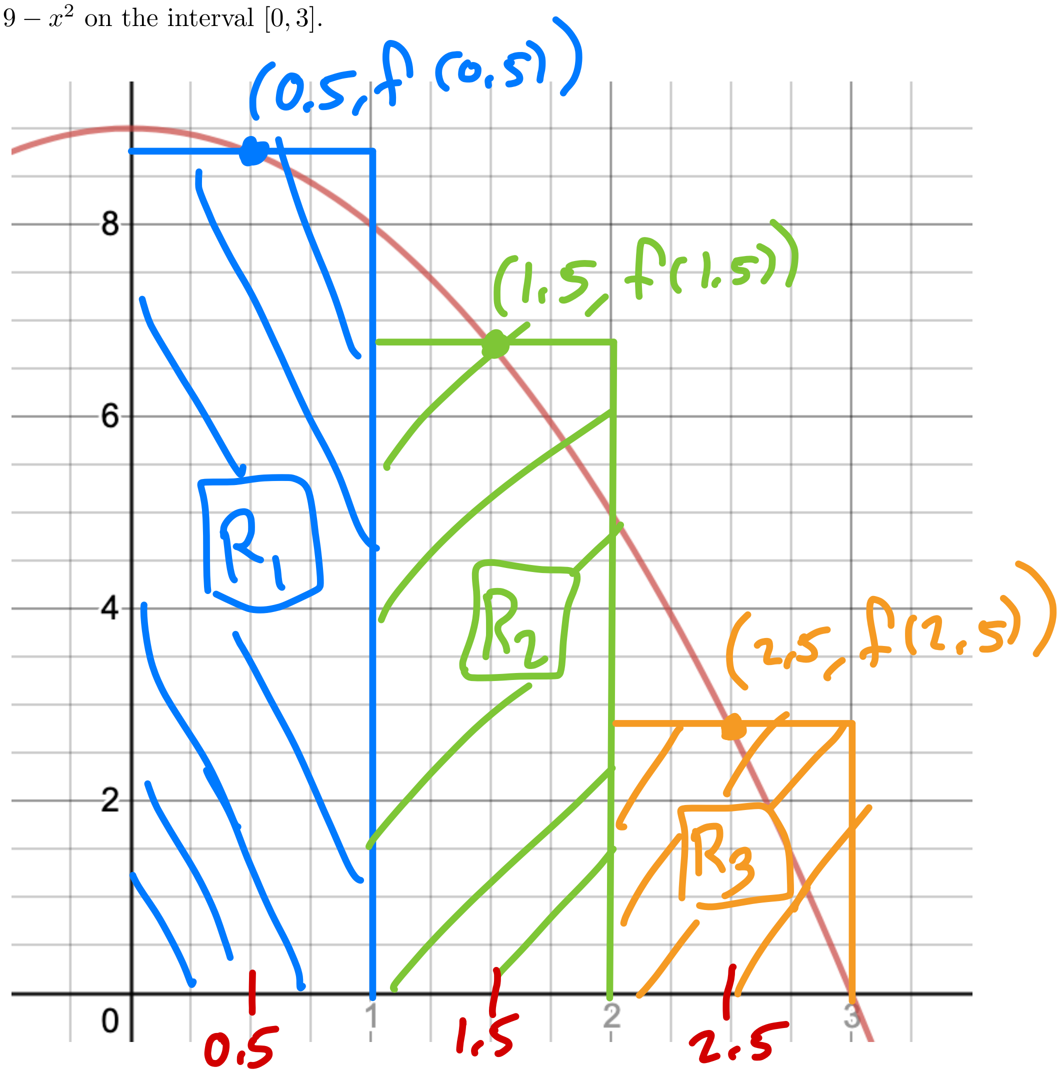
estimated area $= R_1 + R_2 + R_3$

$= 8 + 5 + 0 = 13$

(underestimated)

Let $f(x) = 9 - x^2$ on the interval $[0, 3]$.

(Midpoint)



estimated area $= R_1 + R_2 + R_3$

$= (1)(f(0.5)) + (1)\cdot f(1.5) + (1)\cdot f(2.5)$

$= (1)(9 - 0.5^2) + (1)\cdot(9 - 1.5^2) + (1)\cdot(9 - 2.5^2)$

$= 8.75 + 6.75 + 2.75$

$= 18.25$

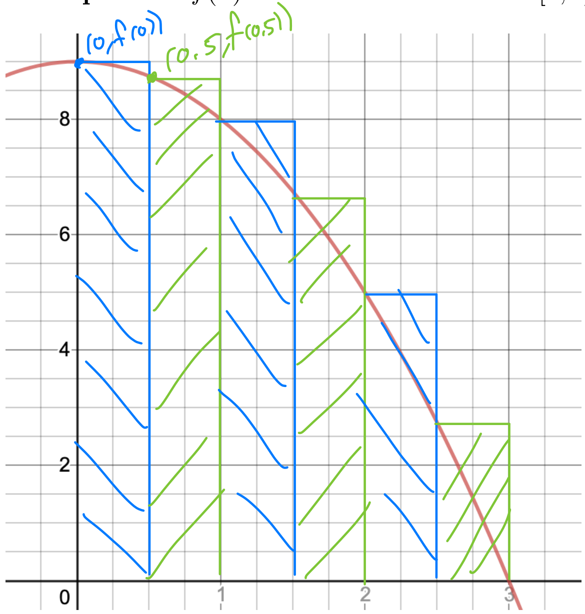
5.1.2 Exact Area

How can we improve our approximation?

Use more rectangles!

Example. Let $f(x) = 9 - x^2$ on the interval $[0, 3]$.

Left



Every rectangle has width

$$\Delta x = \frac{b - a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

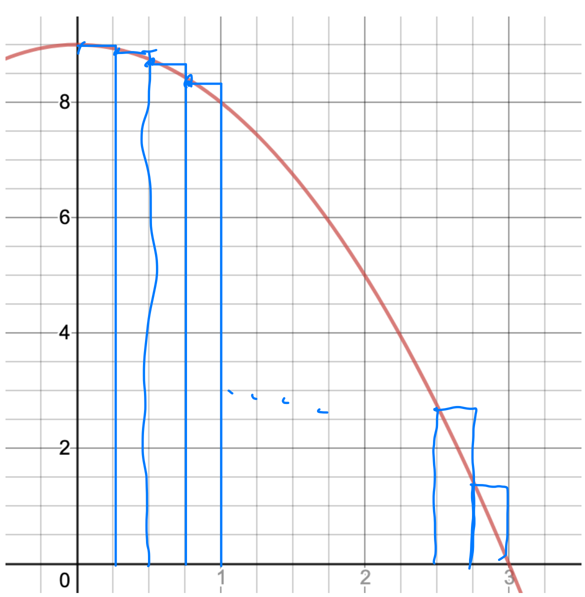
right endpoint, left endpoint, number of rectangles

L_6 (left sum with 6 rectangles) $= \Delta x\,[f(0) + f(0.5) + f(1) + f(1.5) + f(2) + f(2.5)]$

use $\Delta x = \frac{1}{2}$ and $f(x) = 9 - x^2$

$\vdots$

$= 20.125$



$n = 12$ rectangles

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{12} = \frac{1}{4} = 0.25$$

$$L_{12} = \Delta x\,[\underbrace{f(0) + f(0.25) + f(0.5) + \dots + f(2.5) + f(2.75)}_{12 \text{ total terms}}]$$

$\vdots$ $\Delta x = \frac{1}{4}$, $f(x) = 9 - x^2$

$= 19.0938$

Exact answer is 18

$L_3 = 22$, $L_6 = 20.125$, $L_{12} = 19.0938$

- more rectangles gives better approximations
- In general, $\Delta = \frac{b-a}{n}$ is the width of rectangles

Area $\approx \Delta x\,[f(x_1) + f(x_2) + \dots + f(x_n)]$,
where $x_1, x_2, \dots, x_n$ are equally spaced x-values between $x = a$ and $x = b$.

5.2 The Definite Integral

Learning Objectives: After completing this section, we should be able to

- define a definite integral as the limit of a Riemann sum.
- evaluate definite integrals using summation properties.
- apply various properties of definite integrals.

Definition. Let f be a continuous function on $[a, b]$. Then, the net area under the curve of $f(x)$ between $x = a$ and $x = b$ is

$A = \lim_{n \to \infty} \Delta x \left[f(x_1) + f(x_2) + \cdots + f(x_n) \right]$, where $\Delta x = \frac{b-a}{n}$ is the width of each rectangle and $x_1, x_2, \ldots, x_n$ are $\underbrace{\text{arbitrary points}}_{\text{left, right, midpoint}}$ within each of the n rectangles

That is a lot to write, can we be lazier?

Definition. Definite Integral: Let f be a function defined on $[a, b]$.

If $A = \lim_{n \to \infty} \Delta x \left[f(x_1) + f(x_2) + \cdots + f(x_n) \right]$ exists, then this limit is the definite integral from a to b of $f(x)$, and it is notated as

"from a to b"

$$A = \int_{\boxed{a}}^{\boxed{b}} \boxed{f(x)}\, dx$$

"integral"

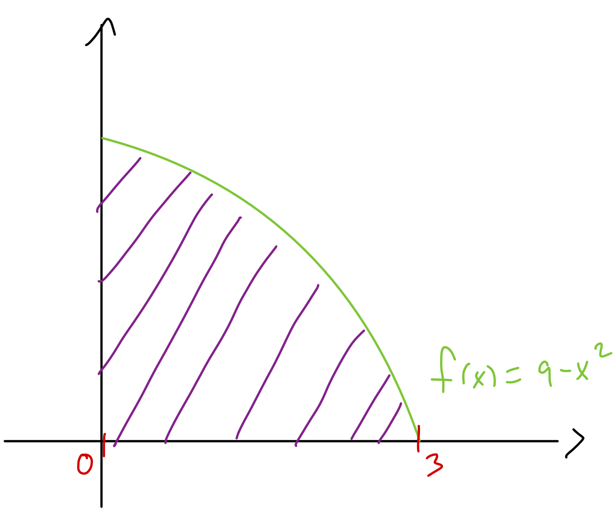
of f(x)dx

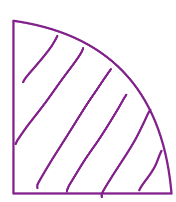
$\Rightarrow$ "The integral from a to b of f(x) dx"

- The number a is the lower limit of integration
- The number b is the upper limit of integration
- If the above limit exists, we say f is integrable on $[a, b]$.

5.2.1 Geometric Understanding of Area

Example. What does $\int_0^3 (9 - x^2)dx$ look like graphically?

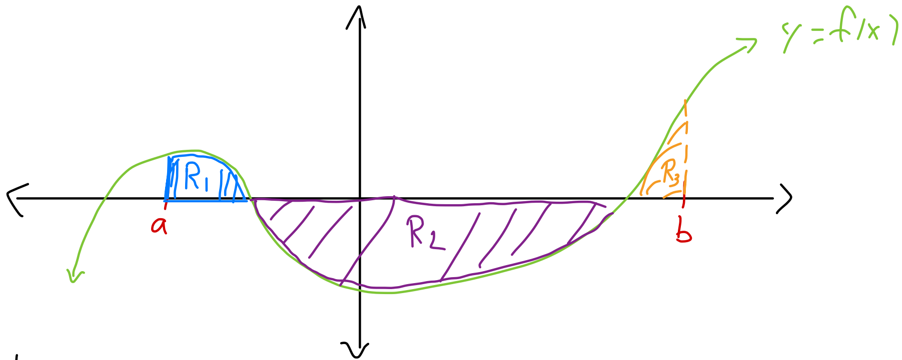


area of  $= \int_0^3 (9 - x^2)\, dx$

Net Area under a Curve

So far, our examples have been positive.

Example.



Net area under f between a and b $= \int_a^b f(x)dx =$ (area above x-axis) − (area below x-axis)

$= (R_1 + R_3) - (R_2)$, assuming we measure the areas R_1, R_2, R_3 in absolute value

Regions below x-axis can "cancel out" with regions above x-axis

We can estimate the area using *Riemann Sums*:

Suppose we have a curve $y = f(x)$ between a and b. If we use n rectangles/intervals to estimate the area under the curve between a & b.

width of rectangles $= \Delta x = \dfrac{b-a}{n}$

- Left Riemann Sum

$$L_n = \Delta x \left[\underbrace{f(a) + f(a+\Delta x) + f(a+2\cdot\Delta x) + \cdots + f(a+(n-1)\cdot\Delta x)}_{n \text{ terms}} \right]$$

- Right Riemann Sum

$b = a + n\cdot\Delta x$

$$R_n = \Delta x \left[\underbrace{f(a+\Delta x) + f(a+2\cdot\Delta x) + \cdots + f(a+n\cdot\Delta x)}_{n \text{ terms}} \right]$$

- Midpoint Riemann Sum

$a + \frac{3\Delta x}{2}$ //

$$M_n = \Delta x \left[\underbrace{f\left(a + \tfrac{\Delta x}{2}\right) + f\left(a + \tfrac{\Delta x}{2} + \Delta x\right) + \cdots + f\left(a + (n-1)\cdot\Delta x + \tfrac{\Delta x}{2}\right)}_{n \text{ terms}} \right]$$

Example. Approximate $\int_{-3}^{1} f(x)dx$ using $n = 4$ rectangles with a Left, Midpoint, and Right Riemman sums.

$$\Delta x = \frac{b-a}{n} = \frac{1-(-3)}{4} = \frac{4}{4} = 1$$

$$L_4 = \Delta x\Big[\underbrace{f(a) + f(a+\Delta x) + f(a+2\cdot\Delta x) + f(a+3\cdot\Delta x)}_{4 \text{ terms}}\Big]$$

$$= 1\cdot[f(-3) + f(-3+1) + f(-3+2\cdot 1) + f(-3+3\cdot 1)]$$

$$= 1\cdot[f(-3) + f(-2) + f(-1) + f(0)]$$

$$R_4 = \Delta x\,[f(a+\Delta x) + f(a+2\cdot\Delta x) + f(a+3\Delta x) + f(a+4\cdot\Delta x)]$$

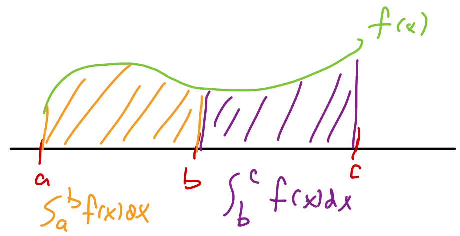
$$= 1\,[f(-3+1) + f(-3+2\cdot 1) + f(-3+3\cdot 1) + f(-3+4\cdot 1)] = 1[f(-2) + f(-1) + f(0) + f(1)]$$

$$M_4 = \Delta x\Big[f\big(a+\tfrac{\Delta x}{2}\big) + f\big(a+\Delta x+\tfrac{\Delta x}{2}\big) + f\big(a+2\Delta x+\tfrac{\Delta x}{2}\big) + f\big(a+3\Delta x+\tfrac{\Delta x}{2}\big)\Big] = 1[f(-2.5) + f(-1.5) + f(-0.5) + f(0.5)]$$

5.2.2 Properties of Definite Integrals

1. $\int_a^b f(x)dx + \int_b^c f(x)dx =$

   $\int_a^c f(x)dx$



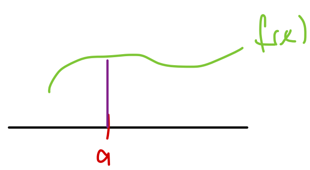
2. $\int_a^b f(x)dx + \int_a^b g(x)dx =$

   $\int_a^b [f(x) + g(x)]dx$

3. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

why? $\Delta x = \frac{b-a}{n}$, then $-\Delta x = -\frac{b-a}{n} = \frac{a-b}{n}$, we let $\Delta x \to dx$

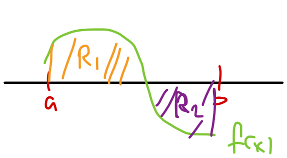
4. $\int_a^a f(x)dx = 0$

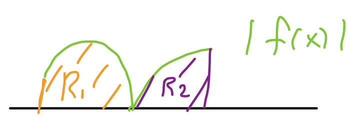


5. $\int_a^b cf(x)dx = c\cdot\int_a^b f(x)dx$,

   where c is any fixed constant.

6. $\int_a^b |f(x)|dx =$ total area between f(x) and x-axis





$2f(x) - g(x) = 2f(x) + (-1)g(x)$

Example. Suppose $\boxed{\int_0^1 f(x)dx = 5}$, $\boxed{\int_0^2 f(x)dx = 2}$, and $\boxed{\int_1^2 g(x)dx = 3}$. What is $\int_1^2 (2f(x) - g(x))dx$?

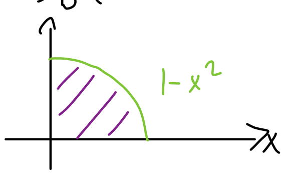
$$\int_1^2 [2\cdot f(x) - g(x)]dx = \int_1^2 2\cdot f(x)dx + \int_1^2 (-1)g(x)dx$$
$$= 2\cdot\int_1^2 f(x)dx + (-1)\int_1^2 g(x)dx$$
$$= 2\int_1^2 f(x)dx + (-1)\cdot 3$$

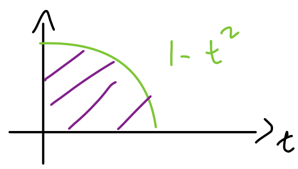
$$\int_0^1 f(x)dx + \int_1^2 f(x)dx = \int_0^2 f(x)dx$$
$$5 + \int_1^2 f(x)dx = 2$$, as $\int_0^1 f(x)dx = 5$ and $\int_0^2 f(x)dx = 2$
$$\Rightarrow \int_1^2 f(x)dx = 2 - 5 = -3$$

$$\Rightarrow = 2(-3) + (-1)(3) = -6 - 3 = -9$$

A note about dummy variables.

$$\int_0^1 (1-x^2)dx = \int_0^1 (1-t^2)dt$$

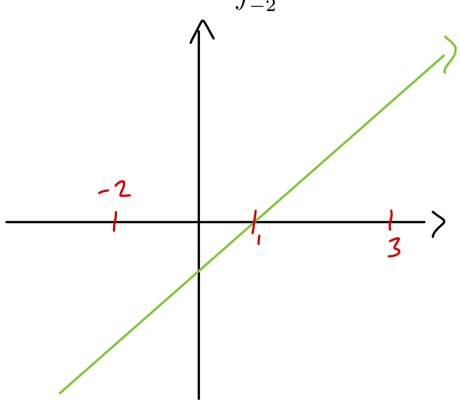




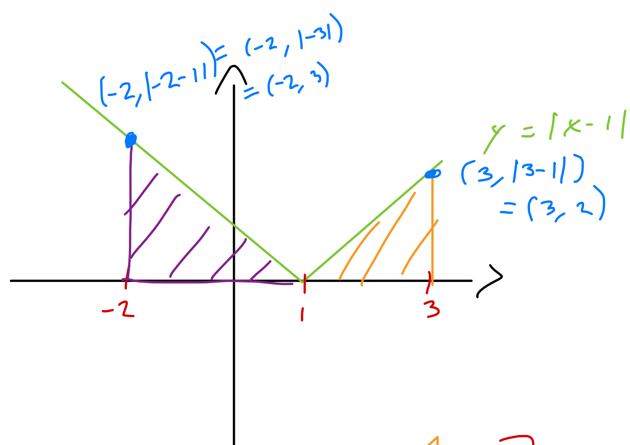
The variable label doesn't matter for definite integrals.

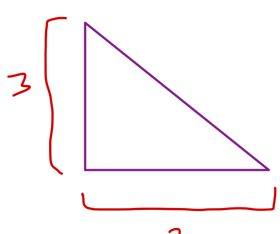
We can compute some definite integrals geometrically.

Example. $\int_{-2}^3 |x-1|dx =$

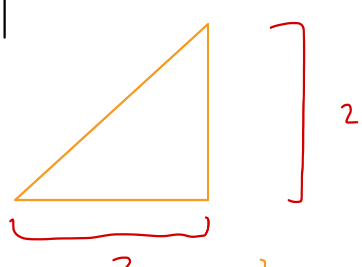


$\Rightarrow$





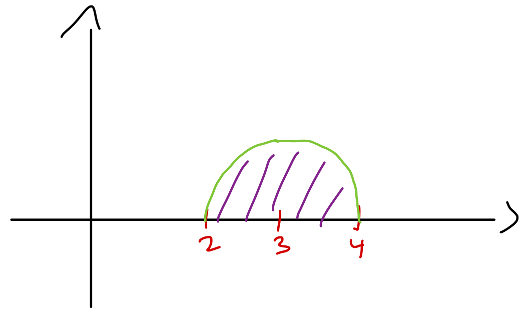
area $= \frac{1}{2} b\cdot h = \frac{1}{2}\cdot 3\cdot 3$

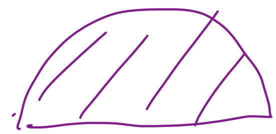


area $= \frac{1}{2} b\cdot h = \frac{1}{2}\cdot 2\cdot 2$

$$\int_{-2}^3 |x-1|dx = \frac{1}{2}\cdot 3\cdot 3 + \frac{1}{2}\cdot 2\cdot 2$$

Example. $\int_2^4 \sqrt{1-(t-3)^2}dt =$





half circle of radius 1

$$A = \frac{1}{2} \cdot \pi \cdot r^2 = \frac{1}{2}\pi \cdot 1^2$$

$$\int_2^4 \sqrt{1-(t-3)^2}\, dt = \frac{1}{2}\pi$$

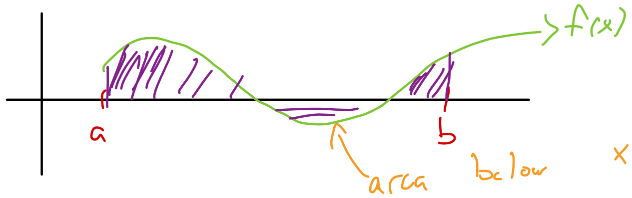
5.3 Fundamental Theorem of Calculus

Learning Objectives: After completing this section, we should be able to

- establish the Fundamental Theorem of Calculus and apply it.
- identify the relationship between differentiation and integration as inverse processes.

5.3.1 Fundamental Theorem of Calculus, part 1

Recall: $\int_a^b f(x)dx =$ net area under f(x) from x = a to x = b



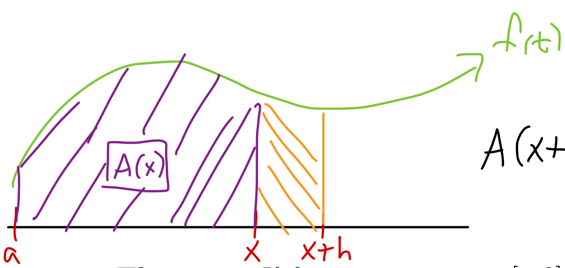
What if we write the function for area under $f(t)$ as
as t ranges from a starting point 'a' to some variable 'x'?

$$A(x) = \int_a^x f(t)dt$$

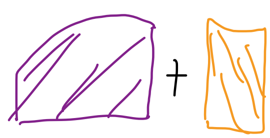
Let's find $\frac{dA}{dx} =$

$$\frac{dA}{dx} = \lim_{h\to 0}\frac{A(x+h)-A(x)}{h} = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h}$$

As $h\to 0$, $\int_x^{x+h} f(t)dt \approx \overset{\text{width}}{h}\cdot \overset{\text{height}}{f(x)}$



$A(x+h) =$



$$= \lim_{h\to 0}\frac{\cancel{h}\cdot f(x)}{\cancel{h}} = f(x)$$

Theorem. *If f is continuous on* $[a, b]$, then, for $a \le x \le b$, we have

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$\boxed{\text{Fundamental Thm of Calculus, part 1}}$

"derivative with respect to x" "of" "the integral from a to x with respect to t of" "f(t)" "is" "f(x)"

Example. $\frac{d}{dx}\int_1^x (t^3+1)dt$

$$= x^3+1$$

5.3.2 Fundamental Theorem of Calculus, part 2

Theorem. *Let f be continuous on $[a, b]$. Then,*

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_{x=a}^{b},$$

shorthand for $F(b) - F(a)$ "evaluate F between a & b".

where $F(x)$ is any antiderivative of $f(x)$; i.e. $F'(x) = f(x)$.

just need 1! we can use $C=0$ if we want

Example. We found an approximate answer for $\int_0^3 (9 - x^2)dx$ in a previous section.

Let $f(x) = 9 - x^2$. Then $F(x) = \int f(x)\,dx$ indefinite integral

$$= \int (9 - x^2)\,dx$$

$x^0 = 1$

$$= 9\int x^0\,dx - \int x^2\,dx$$

$$= 9\frac{1}{0+1}x^{0+1} - \frac{1}{2+1}x^{2+1} + C$$

$$= 9x - \frac{1}{3}x^3 + C$$

We'll use the antiderivative where $C = 0$.

$$\int_0^3 (9 - x^2)dx = F(x)\Big|_{x=0}^{3} = F(3) - F(0)$$

$$= \left(9\cdot 3 - \frac{1}{3}\cdot 3^3\right) - \left(9\cdot 0 - \frac{1}{3}\cdot 0^3\right)$$

okay to stop here

$$= (27 - 9) - (0) = 18$$

Example. Evaluate $\int_5^9 (7x^3 - e^x)dx$.

(we'll be brief)

$$\int_5^9 (7x^3 - e^x)\,dx = \left(7\frac{1}{3+1}x^{3+1} - e^x\right)\Big|_{x=5}^{9}$$

$$= \left(7\frac{1}{3+1}\cdot 9^{3+1} - e^9\right) - \left(7\frac{1}{3+1}\cdot 5^{3+1} - e^5\right)$$

okay to stop here

Example. Evaluate $\int_1^4 \left(\frac{3}{x^2} - 9\right) dx$. $= \int_1^4 (3x^{-2} - 9 \cdot x^0)\, dx$

$$= \left(3 \frac{1}{-2+1} x^{-2+1} - 9 \frac{1}{0+1} x^{0+1}\right)\Big|_{x=1}^{4}$$

$$= \left(3 \cdot \frac{1}{-2+1} \cdot 4^{-2+1} - 9 \frac{1}{0+1} 4^{0+1}\right) - \left(3 \cdot \frac{1}{-2+1} \cdot 1^{-2+1} - 9 \frac{1}{0+1} 1^{0+1}\right)$$

stop here

$$= -\frac{99}{4} = -24.75$$

You try!

Example. Evaluate $\int_1^3 \left(3x^2 - 8x + \frac{1}{x}\right) dx$.

$$= \left(3 \frac{1}{2+1} x^{2+1} - 8 \frac{1}{1+1} x^{1+1} + \ln|x|\right)\Big|_{x=1}^{3}$$

$$= \left(3 \cdot \frac{1}{2+1} 3^{2+1} - 8 \frac{1}{1+1} 3^{1+1} + \ln|3|\right) - \left(3 \cdot \frac{1}{2+1} \cdot 1^{2+1} - 8 \cdot \frac{1}{1+1} \cdot 1^{1+1} + \ln|1|\right)$$

You try!

Example. Evaluate $\int_{\frac{\pi}{2}}^{\pi} \cos(x) dx$.

$$= (\sin(x))\Big|_{x=\frac{\pi}{2}}^{\pi}$$

$$= \sin(\pi) - \sin\left(\frac{\pi}{2}\right)$$

stop here

$$= 0 - 1 = -1$$

$\int e^{x^2} dx$ has no elementary fnctn antiderivative